CS 372: Computational Geometry
Lecture 11
Voronoi Diagrams and Delaunay Triangulations

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Course Information

- Today: Lecture 11.
- Wednesday 14/11: Lecture 12.
- Sunday 18/11: Tutorial.
- Wednesday 21/11: Midterm.

Outline

Today: Geometry, no algorithm.

- Two problems:
  - Nearest neighbor search, mesh generation.
  - Surprisingly, they are closely related.
- Geometric concepts:
  - Voronoi diagram, Delaunay triangulation.
  - Planar graph duality.

Reference:

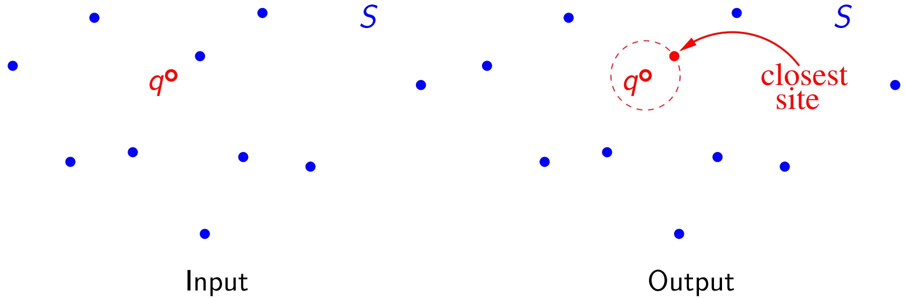
- Textbook Chapter 7 and 9.
- Dave Mount's lecture notes, lectures 16–17.
- Demo by Jack Snoeyink.

Nearest Neighbor Search (NNS)

Definition (Nearest neighbor search in $\mathbb{R}^2$)

Preprocess a set S of n points in $\mathbb{R}^2$ so as to be able to answer the following queries efficiently.

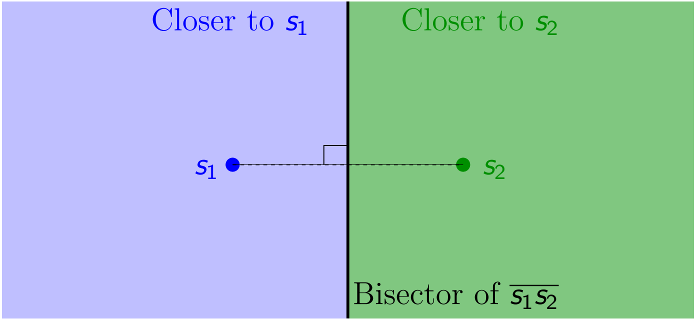
- Query: point $q \in \mathbb{R}^2$.
- Output: point $s \in S$ that is closest to q.



Nearest Neighbor Search (NNS)

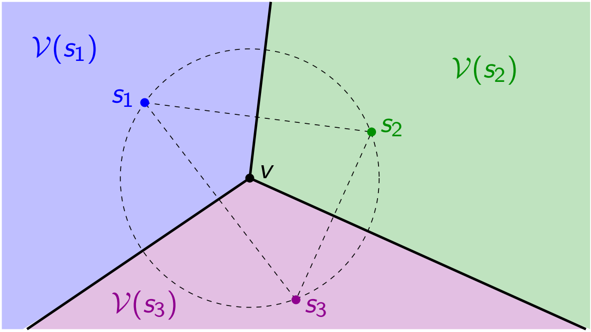
Approach:

- Draw a diagram.
- Example with $|S| = 2$:



- This is the *Voronoi diagram* of $S = \{s_1, s_2\}$.

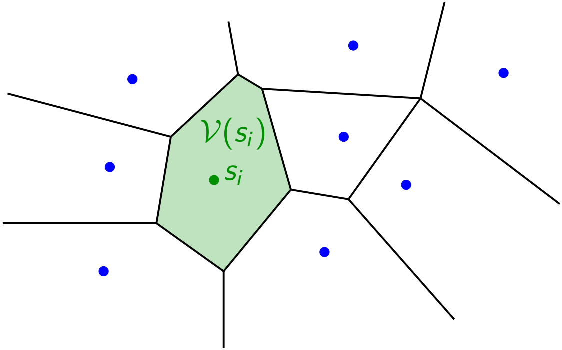
Example with $|S| = 3$



The Voronoi diagram of $S = \{s_1, s_2, s_3\}$ consists of:

- The *Voronoi vertex* v, which is the center of the circumcircle of triangle $s_1 s_2 s_3$.
- The *Voronoi cells* $\mathcal{V}(s_1)$, $\mathcal{V}(s_2)$, $\mathcal{V}(s_3)$.
- 3 *Voronoi edges*.

Example

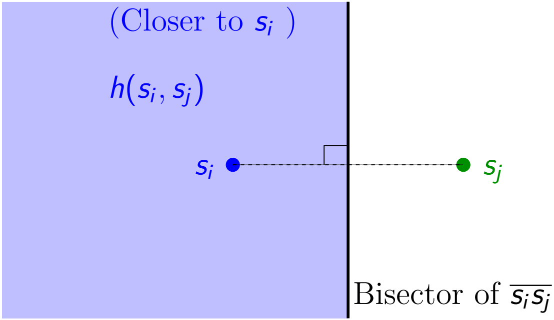


$$\mathcal{V}(s_i) = \{x \in \mathbb{R}^2 \mid \forall j \neq i, \ |s_i x| < |s_j x|\}$$

Half-Plane $h(s_i, s_j)$

Notation:

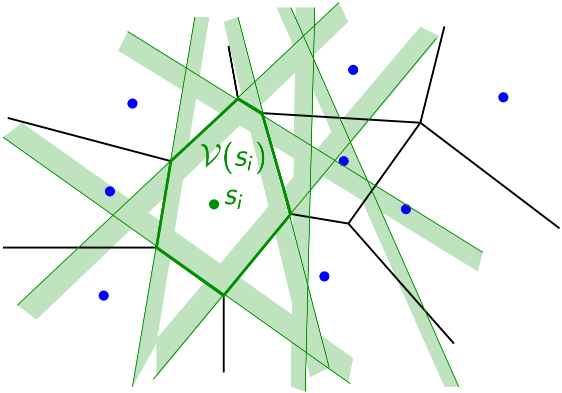
$$h(s_i, s_j) = \left\{ x \in \mathbb{R}^2 \middle| |s_i x| < |s_j x| \right\}$$



Voronoi Cell

It follows that

$$\mathcal{V}(s_i) = \bigcap_{j \neq i} h(s_i, s_j).$$



Properties

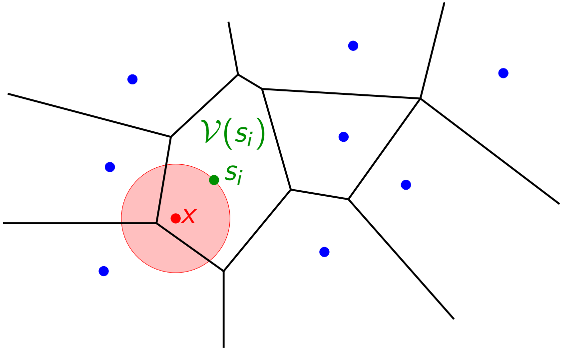
- The Voronoi diagram of S is *not* a planar straight line graph.
  - Reason: It has infinite edges.
  - To fix this problem, we can restrict our attention to the portion of the Voronoi diagram that is within a large bounding box.
- All the cells are convex, hence connected.
- So the Voronoi diagram has n faces, one for each site.
- It has $O(n)$ edges and vertices.
  - Non trivial. Follows from:
    - ⋆ Vertices have degree at least 3,
    - ⋆ Double counting,
    - ⋆ and Euler's relation.

Algorithmic Consequences

- $\mathcal{V}(s_i)$ is an intersection of n half-planes.
- So it can be computed in $O(n \log n)$ time.
- We can compute the Voronoi diagram of S in $O(n^2 \log n)$ time.
- We associate it with a point location data structure.
- So we can do nearest neighbor searching in:
  - $O(n^2 \log n)$ preprocessing time.
  - Expected $O(n)$ space usage.
  - Expected $O(\log n)$ query time.
- Next lecture: preprocessing time down to expected $\Theta(n \log n)$

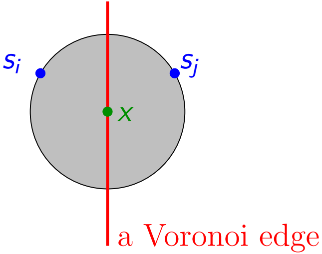
Voronoi Cell

$\forall x \in \mathcal{V}(s_i)$, the disk through s_i centered at x contains no other site than s_i.



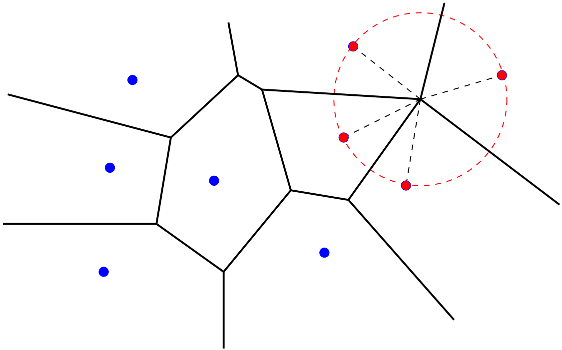
Voronoi Edges

- A *Voronoi edge* is an edge of the Voronoi diagram.
- A point x on a Voronoi edge is equidistant to two nearest sites s_i and s_j.
- Hence the circle centered at x through s_i and s_j contains no site in its interior.



General Position Assumption

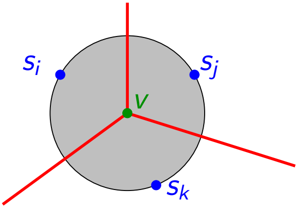
General position assumption: No four sites are cocircular.



A degenerate case: 4 sites lie on the same circle

Voronoi Vertices

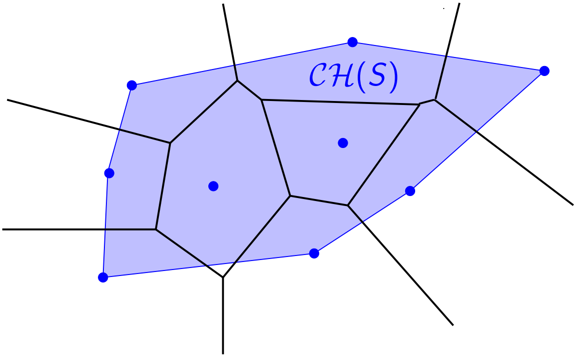
- A Voronoi vertex v is equidistant to three nearest sites s_i, s_j and s_k.
- Hence the circle centered at v through s_i, s_j and s_k contains no site in its interior.



- By our general position assumption, each Voronoi vertex has degree 3.

Voronoi Cells

- If $\mathcal{V}(s_i)$ is bounded, then it is a convex polygon.
- $\mathcal{V}(s_i)$ is unbounded iff s_i is a vertex of $\mathcal{CH}(S)$.



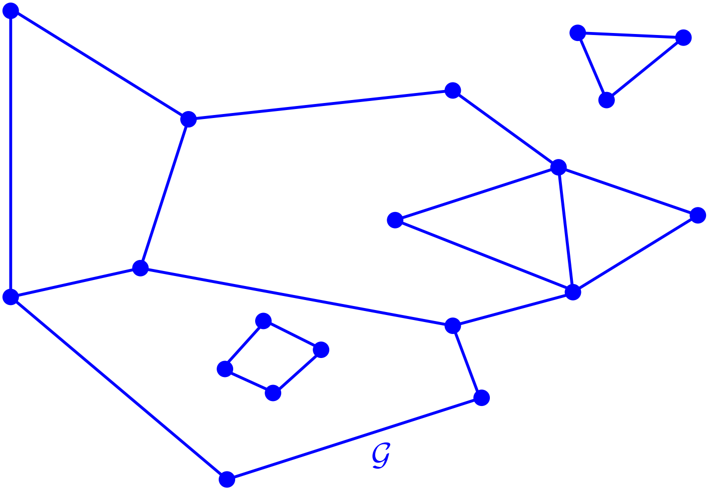
Consequence

- Knowing the Voronoi diagram, we can compute the convex hull in $O(n)$ time.
- So computing a Voronoi diagram takes $\Omega(n \log n)$ time.
- Next lecture: An optimal $O(n \log n)$ time randomized algorithm.
- There is also a deterministic $O(n \log n)$ time algorithm.
  - Plane-sweep algorithm.
  - Lecture 16 of D. Mount or Lecture 7 in textbook.
  - Not covered in CS 372.

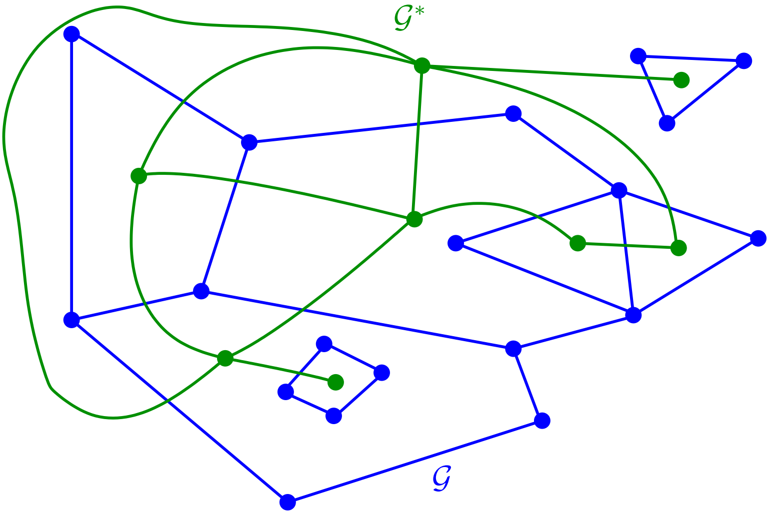
Further Remarks

- Sites need not be points: one can define in the same way the Voronoi diagram of a set of line segments, or any shape.
- We can also use different distance functions.
- The Voronoi diagram can also be defined in $\mathbb{R}^d$ for any d.
  - But it has size $\Theta\left(n^{\lceil d/2 \rceil}\right)$.
  - So it is only useful when d is small (say, at most 4).

Dual of a Planar Graph



Dual of a Planar Graph



Dual of a Planar Graph

Definition

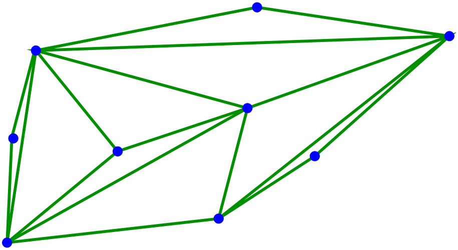
Let $\mathcal{G}$ be a planar graph, embedded in the plane. The *dual graph* $\mathcal{G}^*$ is such that:

- Each vertex f^* of $\mathcal{G}^*$ corresponds to a face f of $\mathcal{G}$
- (f^*, g^*) is an edge of $\mathcal{G}^*$ iff f and g are adjacent in $\mathcal{G}$.

Property

The dual of a planar graph is planar.

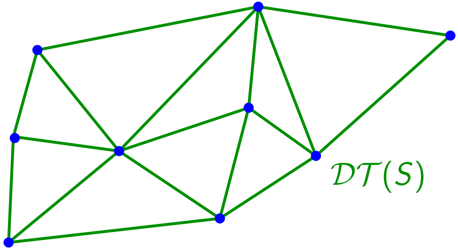
Triangulation of a point-set



Definition (Point-set triangulation)

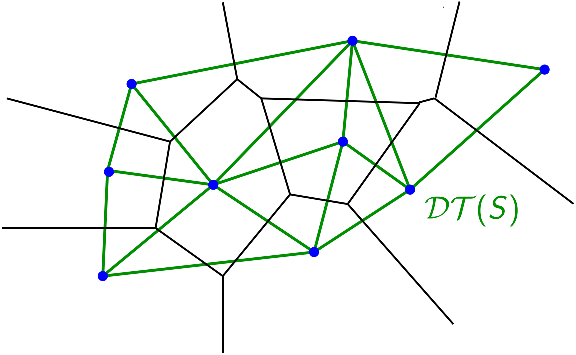
Given a set S of n points in $\mathbb{R}^2$, a *triangulation* of S is a planar graph with vertex set S, such that all the bounded faces are triangles, and these faces form a partition of the convex hull $\mathcal{CH}(S)$ of S.

The Delaunay Triangulation



- The *Delaunay triangulation* of the same set.
- It has many interesting properties.

The Delaunay Triangulation



The Delaunay Triangulation

Definition (Delaunay triangulation)

Let S be a set of n points in $\mathbb{R}^2$. We assume general position in the sense that no 4 points in S are cocircular. The *Delaunay triangulation* $\mathcal{DT}(S)$ of S is the dual graph of the Voronoi diagram of S such that:

- Each vertex $\mathcal{V}(s_i)^*$ is located at the corresponding site s_i.
- The edges of $\mathcal{DT}(S)$ are straight line segments.

Remarks

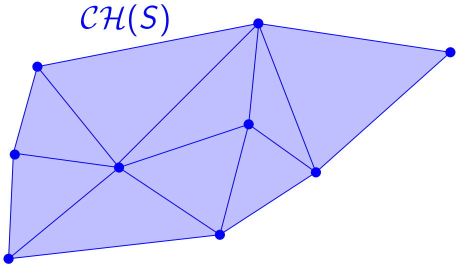
Is $\mathcal{DT}(S)$ well defined?

We need to prove that:

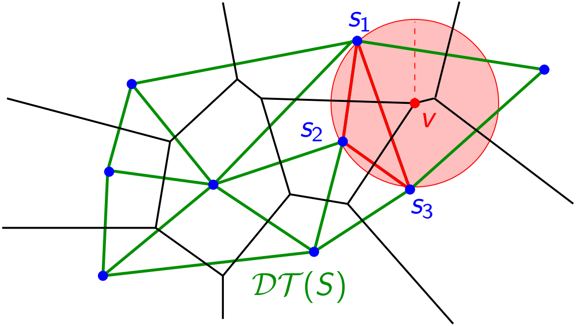
- Edges do not intersect ($=$ it is a PSLG).
  - Left as an exercise.
- Faces are triangles.
  - The number of edges in a face of $\mathcal{DT}(S)$ is the degree of the corresponding Voronoi vertex.
  - General position assumption implies that Voronoi vertices have degree 3.

Convex Hull

- The convex hull of S is the complement of the unbounded face of $\mathcal{DT}(S)$.



Circumcircle Property

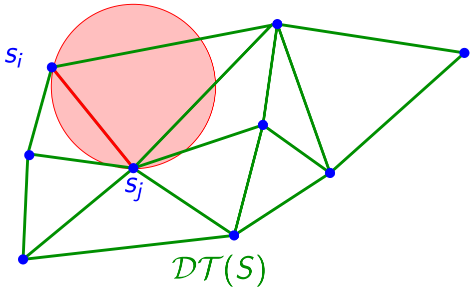


Property (Circumcircle)

The circumcircle of any triangle in $\mathcal{DT}(S)$ is empty. (It contains no site s_i in its interior.)

Proof: Let $s_1 s_2 s_3$ be a triangle in $\mathcal{DT}(S)$, let v be the corresponding Voronoi vertex. Property of Voronoi vertices: the circle centered at v through $s_1 s_2 s_3$ is empty.

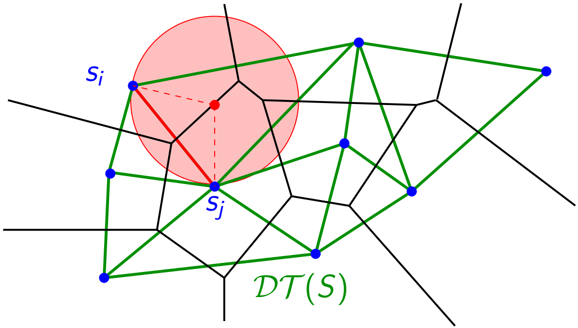
Empty Circle Property



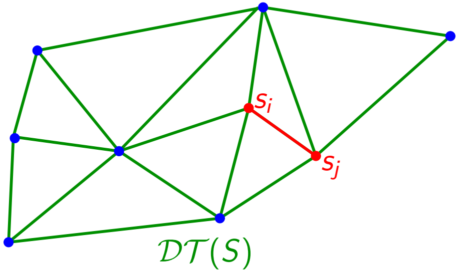
Property (Empty circle)

$\overline{s_i s_j}$ is an edge of $\mathcal{DT}(S)$ iff there is an empty circle through s_i and s_j.

Proof (Empty Circle Property)



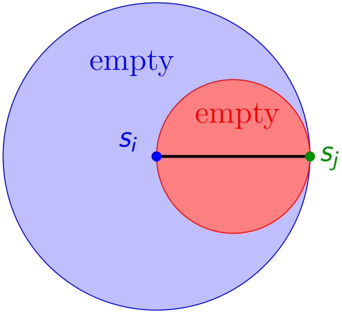
Closest Pair Property



Property (Closest pair)

The closest two sites $s_i s_j$ are connected by an edge of $\mathcal{DT}(S)$.

Proof (Closest Pair Property)

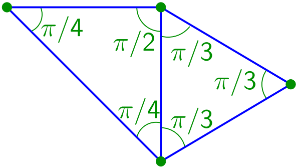


Euclidean Minimum Spanning Tree

- Euclidean graph:
  - Set of vertices $= S$
  - For any $i \neq j$, there is an edge between s_i and s_j with weight $|s_i s_j|$.
- Euclidean Minimum Spanning Tree (EMST): Minimum spanning tree of the Euclidean graph.
- Property: The EMST is a subgraph of $\mathcal{DT}(S)$.
- Corollary: It can be computed in $O(n \log n)$ time.
- See D. Mount's notes pages 75–76.

Angle Sequence

- Let $\mathcal{T}$ be a triangulation of S.
- Angle sequence $\Theta(\mathcal{T})$: Sequence of all the angles of the triangles of $\mathcal{T}$ in non-decreasing order.
- Example:



$$\Theta(\mathcal{T}) = (\pi/4, \pi/4, \pi/3, \pi/3, \pi/3, \pi/2)$$

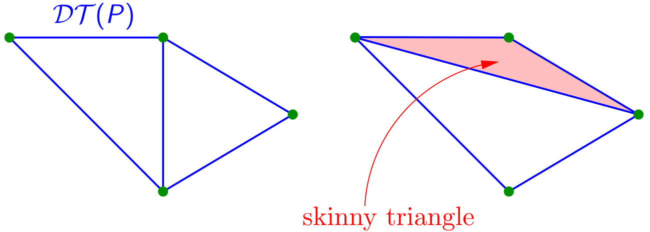
- Comparison: Let $\mathcal{T}$ and $\mathcal{T}'$ be two triangulations of S. We compare $\Theta(\mathcal{T})$ and $\Theta(\mathcal{T}')$ using *lexicographic order*.
- Example: $(1, 1, 3, 4, 5) < (1, 2, 4, 4, 4)$.

Optimality of the Delaunay Triangulation $\mathcal{DT}(S)$

Theorem

Let S be a set of points in general position. Then the angle sequence of $\mathcal{DT}(S)$ is maximal among all triangulations of S.

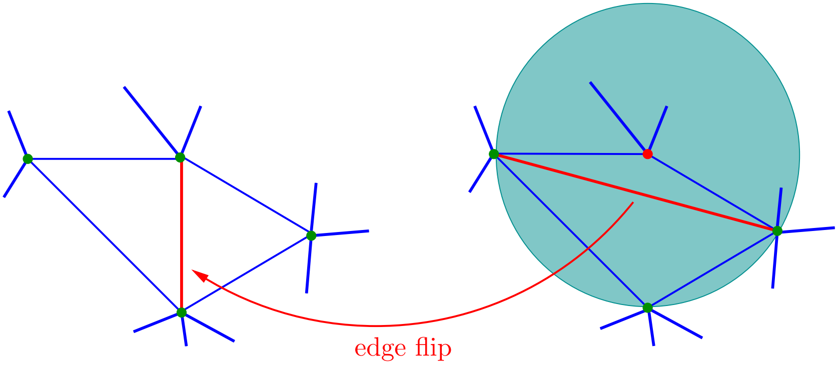
- So the Delaunay triangulation maximizes the minimum angle.
- Intuition: Avoids skinny triangles.



Proof

Idea:

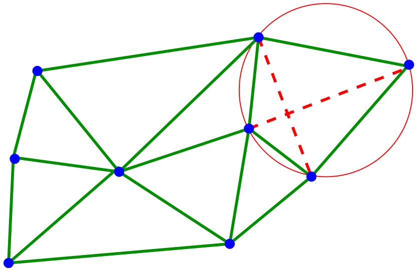
- Flip edges to ensure the circumcircle property.
- It increases the angle sequence.



Degenerate Cases

Several possible Delaunay triangulations.

Example:



Two possibilities.

Any Delaunay triangulation maximizes the minimum angle. But the angle sequences of two Delaunay triangulations may differ.